Web-based Supplementary Materials for "A mixed effects Bayesian regression model for multivariate group testing data"

Web Appendix A: Full conditional distributions

The full conditional distributions used to construct our posterior sampling algorithm are given below:

$$\begin{aligned}
\widetilde{Y}_{id} \mid \widetilde{\mathbf{Y}}_{i(-d)}, \mathbf{Z}, \boldsymbol{\Theta} &\sim \text{Bernoulli}(p^*_{id}), \\
\boldsymbol{\omega}_i \mid \widetilde{\mathbf{Y}}_i, \boldsymbol{\beta}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}_{(i)}, \mathbf{R} &\sim TN(\boldsymbol{\eta}_i, \mathbf{R}, \mathbf{L}_i, \mathbf{U}_i), \\
\boldsymbol{\beta}_{\boldsymbol{v}} \mid \boldsymbol{\omega}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v} &\sim N(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}}), \\
\lambda_{ld} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, \mathbf{R}, w_{ld} &\sim TN\{\mu_{\lambda_{ld}} w_{ld}, \sigma^2_{\lambda_{ld}} w_{ld}, 0, \infty\}, \\
\mathbf{a} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \mathbf{b}, \mathbf{R} &\sim N(\boldsymbol{\mu}_{\mathbf{a}}, \boldsymbol{\Sigma}_{\mathbf{a}}) \\
\mathbf{b}_k \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{R} &\sim N(\boldsymbol{\mu}_{\mathbf{b}_k}, \boldsymbol{\Sigma}_{\mathbf{b}_k}), \\
v_{rd} \mid \boldsymbol{\omega}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}_{(-rd)}, \tau_{v_{rd}} &\sim \text{Bernoulli}(p_{v_{rd}}), \\
w_{ld} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, \tau_{w_{ld}} &\sim \text{Bernoulli}(p_{w_{ld}}), \\
\tau_{v_{rd}} | v_{rd} &\sim \text{Beta}(a_v + v_{rd}, b_v + 1 - v_{rd}), \\
\tau_{w_{ld}} | w_{ld} &\sim \text{Beta}(a_w + w_{rd}, b_w + 1 - w_{rd}), \\
S_{e(m):d} \mid \mathbf{Z}, \widetilde{\mathbf{Y}} &\sim \text{Beta}(a^\star_{e(m):d}, b^\star_{e(m):d}), \\
S_{p(m):d} \mid \mathbf{Z}, \widetilde{\mathbf{Y}} &\sim \text{Beta}(a^\star_{p(m):d}, b^\star_{p(m):d}),
\end{aligned}$$

where the specific form of the parameters of these distribution are provided below. To present these specific forms, we make use of the following notation: $\mathbf{X}_i = \oplus_{d=1}^{D} \mathbf{x}'_{id}$, $\mathbf{T}_i = \oplus_{d=1}^{D} \mathbf{t}'_{id}$, $\boldsymbol{\Lambda} = \oplus_{d=1}^{D} \boldsymbol{\Lambda}_d$, $\mathbf{A} = \oplus_{d=1}^{D} \mathbf{A}_d$, $\boldsymbol{v} = (\boldsymbol{v}'_1, ..., \boldsymbol{v}'_D)'$, and $\boldsymbol{v}_d = (v_{1d}, ..., v_{p_d d})'$

Full conditional of $\widetilde{Y}_{id}$: From the joint distribution of the observed testing outcomes and the individuals' latent statuses, which is given by

$$\pi(\mathbf{Z},\widetilde{\mathbf{Y}}\mid\boldsymbol{\Theta})=\prod_{d=1}^{D}\prod_{m=1}^{M}\prod_{j\in\mathcal{I}_m}\left\{S_{e(m):d}^{Z_{jd}}(1-S_{e(m):d})^{1-Z_{jd}}\right\}^{\widetilde{Z}_{jd}}\left\{S_{p(m):d}^{1-Z_{jd}}(1-S_{p(m):d})^{Z_{jd}}\right\}^{1-\widetilde{Z}_{jd}}$$
$$\times\prod_{i=1}^{N}\pi(\widetilde{\mathbf{Y}}_i\mid\boldsymbol{\beta},\boldsymbol{\lambda},\mathbf{a},\mathbf{b}_{(i)},\mathbf{R}),$$

it is relatively easy to see that the full conditional distribution of $\widetilde{Y}_{id}$ is Bernoulli. In particular, $\widetilde{Y}_{id}\mid\widetilde{\mathbf{Y}}_{i(-d)},\mathbf{Z},\boldsymbol{\Theta}\sim\text{Bernoulli}(p_{id}^{*})$, where $\widetilde{\mathbf{Y}}_{i(-d)}$ is the vector $\widetilde{\mathbf{Y}}_i$ with the dth element removed, $p_{id}^{*}=p_{id1}^{\star}/(p_{id0}^{\star}+p_{id1}^{\star})$, and

$$p_{id1}^{\star}=p_{id}\prod_{j\in\mathcal{A}_i}S_{e_j:d}^{Z_{jd}}(1-S_{e_j:d})^{1-Z_{jd}}$$
$$p_{id0}^{\star}=(1-p_{id})\prod_{j\in\mathcal{A}_i}\left\{S_{e_j:d}^{Z_{jd}}(1-S_{e_j:d})^{1-Z_{jd}}\right\}^{I\left(s_{ijd}>0\right)}\left\{(1-S_{p_j:d})^{Z_{jd}}S_{p_j:d}^{1-Z_{jd}}\right\}^{I(s_{ijd}=0)}.$$

In the expression above $p_{id}=\pi(\widetilde{\mathbf{Y}}_{i(d)}\mid\boldsymbol{\beta},\boldsymbol{\lambda},\mathbf{a},\mathbf{b}_{(i)},\mathbf{R})$, $\widetilde{\mathbf{Y}}_{i(d)}=(\widetilde{Y}_{i1},...,\widetilde{Y}_{id}=1,...,\widetilde{Y}_{iD})'$, the index set $\mathcal{A}_i=\{j\colon i\in\mathcal{P}_j\}$ keeps track of which pools the ith individual was a member of, $s_{ijd}=\sum_{i'\in\mathcal{P}_j\colon i'\neq i}\widetilde{Y}_{i'd}$, and if $j\in\mathcal{I}_m$ then $S_{e_j:d}=S_{e(m):d}$ and $S_{p_j:d}=S_{p(m):d}$.

Full conditional of $\boldsymbol{\omega}_i$: By inspecting the following joint distribution

$$\pi(\mathbf{Z},\widetilde{\mathbf{Y}},\boldsymbol{\omega}\mid\boldsymbol{\Theta})\propto\prod_{d=1}^{D}\prod_{m=1}^{M}\prod_{j\in\mathcal{I}_m}\left\{S_{e(m):d}^{Z_{jd}}(1-S_{e(m):d})^{1-Z_{jd}}\right\}^{\widetilde{Z}_{jd}}\left\{S_{p(m):d}^{1-Z_{jd}}(1-S_{p(m):d})^{Z_{jd}}\right\}^{1-\widetilde{Z}_{jd}}$$
$$\times\prod_{i=1}^{N}|\mathbf{R}|^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{\omega}_i-\boldsymbol{\eta}_i)'\mathbf{R}^{-1}(\boldsymbol{\omega}_i-\boldsymbol{\eta}_i)\right\}\prod_{i=1}^{N}f(\boldsymbol{\omega}_i),$$

one can easily see that the full conditional distribution of $\boldsymbol{\omega}_i$ is multivariate truncated normal with mean $\boldsymbol{\eta}_i$, covariance matrix $\mathbf{R}$, lower truncation limits $\mathbf{L}_i=(L_{i1},...,L_{iD})'$, and upper truncation limits $\mathbf{U}_i=(U_{i1},...,U_{iD})'$, such that the truncation region for the dth dimension is given by $L_{id}=0$ and $U_{id}=\infty$ if $\widetilde{Y}_{id}=1$ and by $L_{id}=-\infty$ and $U_{id}=0$ if $\widetilde{Y}_{id}=0$; i.e.,

$$\boldsymbol{\omega}_i\mid\widetilde{\mathbf{Y}}_i,\boldsymbol{\beta},\boldsymbol{\lambda},\mathbf{a},\mathbf{b}_{(i)},\mathbf{R}\sim TN\{\boldsymbol{\eta}_i,\mathbf{R},\mathbf{L}_i,\mathbf{U}_i\}.$$

Full conditional of $\boldsymbol{\beta}$: The full conditional distribution of β_{rd} is degenerate at 0 if $v_{rd}=0$, while the nonzero elements of $\boldsymbol{\beta}$, say $\boldsymbol{\beta}_{\boldsymbol{v}}$, have the following normal full conditional distribution

$$\boldsymbol{\beta}_{\boldsymbol{v}}\mid\boldsymbol{\omega},\boldsymbol{\lambda},\mathbf{a},\mathbf{b},\mathbf{R},\boldsymbol{v},\sim N(\boldsymbol{\mu}_{\boldsymbol{\beta}},\boldsymbol{\Sigma}_{\boldsymbol{\beta}}),$$

where the mean and covariance matrix are

$$\boldsymbol{\mu}_{\boldsymbol{\beta}} = \left(\boldsymbol{\Phi}(\boldsymbol{v})^{-1} + \sum_{i=1}^{N} \mathbf{X}_i(\boldsymbol{v})'\mathbf{R}^{-1}\mathbf{X}_i(\boldsymbol{v})\right)^{-1} \times \sum_{i=1}^{N} \mathbf{X}_i(\boldsymbol{v})'\mathbf{R}^{-1}\boldsymbol{\omega}^{\star}_{\boldsymbol{\beta}i}$$
$$\boldsymbol{\Sigma}_{\boldsymbol{\beta}} = \left(\boldsymbol{\Phi}(\boldsymbol{v})^{-1} + \sum_{i=1}^{N} \mathbf{X}_i(\boldsymbol{v})'\mathbf{R}^{-1}\mathbf{X}_i(\boldsymbol{v})\right)^{-1},$$

and $\boldsymbol{\Phi} = \mathrm{diag}(\phi^2_{rd}; r = 1, ..., p_d, d = 1, ..., D)$, $\boldsymbol{\Phi}(\boldsymbol{v})$ is the matrix that is formed by retaining the rows and columns of $\boldsymbol{\Phi}$ that correspond to the non-zero elements of $\boldsymbol{v}$, $\mathbf{X}_i(\boldsymbol{v})$ is the matrix that is formed by retaining the columns of $\mathbf{X}_i$ corresponding to the non-zero elements of $\boldsymbol{v}$, and $\boldsymbol{\omega}^{\star}_{\boldsymbol{\beta}i} = \boldsymbol{\omega}_i - \mathbf{T}_i\boldsymbol{\Lambda}\mathbf{A}\mathbf{b}_{(i)}$.

Full conditional of λ_{ld}: To present the full conditional distribution of λ_{ld}, we first introduce a new set of notation. For the ith individual define a $q_d \times 1$ vector $\mathbf{e}_{id}$ whose lth element is $t_{idl}b_{(i)dl} + t_{idl}\sum_{m=1}^{l-1} b_{(i)dm}a_{dlm}$, where t_{idl} is the lth element of $\mathbf{t}_{id}$, $b_{(i)dl}$ is the lth element of $\mathbf{b}_{(i)d}$, and a_{dlm} is the (l, m)th entry of $\mathbf{A}_d$. Construct $\mathbf{E}_i = \oplus_{d=1}^{D}\mathbf{e}'_{id}$. Based on this new notation, we can succinctly express the full conditional distribution of λ_{ld}, which is the ℓth element of $\boldsymbol{\lambda}$. In particular, the full conditional distribution of λ_{ld} is degenerate at 0 if $w_{ld} = 0$, and when $w_{ld} = 1$ the full conditional is given by

$$\lambda_{ld} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, \mathbf{R}, w_{ld} \sim TN\{\mu_{\lambda_{ld}}, \sigma^2_{\lambda_{ld}}, 0, \infty\},$$

where the mean and variance are

$$\mu_{\lambda_{ld}} = \left(1/\boldsymbol{\Psi}_{\ell\ell} + \sum_{i=1}^{N} \mathbf{E}^{\ell\prime}_i\mathbf{R}^{-1}\mathbf{E}^{\ell}_i\right)^{-1} \times \sum_{i=1}^{N} \mathbf{E}^{\ell\prime}_i\mathbf{R}^{-1}\boldsymbol{\omega}^{\star}_{\lambda_\ell i}$$
$$\sigma^2_{\lambda_{ld}} = \left(1/\boldsymbol{\Psi}_{\ell\ell} + \sum_{i=1}^{N} \mathbf{E}^{\ell\prime}_i\mathbf{R}^{-1}\mathbf{E}^{\ell}_i\right)^{-1}.$$

In the expressions above $\mathbf{E}^{\ell}_i$ denotes the ℓth column of $\mathbf{E}_i$, $\boldsymbol{\Psi}_{\ell\ell}$ is the ℓth diagonal element of $\boldsymbol{\Psi} = \mathrm{diag}(\psi^2_{ld}; l = 1, ..., q_d, d = 1, ..., D)$, $\boldsymbol{\omega}^{\star}_{\lambda_\ell i} = \boldsymbol{\omega}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{E}^{(\ell)}_i\boldsymbol{\lambda}_{(-\ell)}$, $\mathbf{E}^{(-\ell)}_i$ is the matrix that remains after removing the ℓth column of $\mathbf{E}_i$, and $\boldsymbol{\lambda}_{(-\ell)}$ is the vector that remains after removing λ_{ld} from $\boldsymbol{\lambda}$.

Full conditional of $\mathbf{a}$: To present the full conditional distribution of $\mathbf{a}$, we first introduce a new set of notation. Define the $q_d \times (q_d - 1)/2$ vector $\mathbf{u}_{id} = (b_{(i)dl}\lambda_{dm}t_{idm}; l = 1, ..., q_d - 1, m =$

$l+1, ..., q_d)'$ and construct $\mathbf{U}_i = \oplus_{d=1}^{D}\mathbf{u}'_{id}$, where $b_{(i)dl}$ is the lth element of $\mathbf{b}_{(i)d}$, λ_{dm} is the mth element of $\boldsymbol{\lambda}_d$, and t_{idm} is the mth element of $\mathbf{t}_{id}$. The linear predictor of our model can then be re-expressed as

$$\eta_{id} = \mathbf{x}'_{id}\boldsymbol{\beta} + \mathbf{t}'_{id}\boldsymbol{\Lambda}_d\mathbf{b}_{(i)d} + \mathbf{u}'_{id}\mathbf{a}_d.$$

Given this observation it is easy to see that the full conditional distribution of $\mathbf{a}$ is given by

$$\mathbf{a} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \mathbf{b}, \mathbf{R} \sim N(\boldsymbol{\mu}_{\mathbf{a}}, \boldsymbol{\Sigma}_{\mathbf{a}})$$

where the mean and covariance matrix are

$$\begin{aligned}\boldsymbol{\mu}_{\mathbf{a}} &= \left(\mathbf{C}^{-1} + \sum_{i=1}^{N}\mathbf{U}'_i\mathbf{R}^{-1}\mathbf{U}_i\right)^{-1} \times \left(\mathbf{C}^{-1}\mathbf{m} + \sum_{i=1}^{N}\mathbf{U}'_i\mathbf{R}^{-1}\boldsymbol{\omega}^{\star}_{\mathbf{a}i}\right)\\ \boldsymbol{\Sigma}_{\mathbf{a}} &= \left(\mathbf{C}^{-1} + \sum_{i=1}^{N}\mathbf{U}'_i\mathbf{R}^{-1}\mathbf{U}_i\right)^{-1},\end{aligned}$$

and $\boldsymbol{\omega}^{\star}_{\mathbf{a}i} = \boldsymbol{\omega}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{T}_i\boldsymbol{\Lambda}\mathbf{b}_{(i)}$, $\mathbf{C} = \mathrm{diag}(\mathbf{C}_1, ..., \mathbf{C}_D)$, and $\mathbf{m} = (\mathbf{m}_1, ..., \mathbf{m}_D)'$.

Full conditional of $\mathbf{b}_k$: Define the index set $\mathcal{S}_k = \{i : \mathbf{b}_{(i)} = \mathbf{b}_k\}$; i.e., the index set of individuals who visited site k. Then the full conditional distribution of $\mathbf{b}_k$ is given by

$$\mathbf{b}_k \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{R} \sim N(\boldsymbol{\mu}_{\mathbf{b}_k}, \boldsymbol{\Sigma}_{\mathbf{b}_k}),$$

where the mean and covariance matrix are

$$\begin{aligned}\boldsymbol{\mu}_{\mathbf{b}_k} &= \left(\mathbf{I} + \sum_{i\in\mathcal{S}_k}\mathbf{A}'\boldsymbol{\Lambda}\mathbf{T}'_i\mathbf{R}^{-1}\mathbf{T}_i\boldsymbol{\Lambda}\mathbf{A}\right)^{-1} \times \sum_{i\in\mathcal{S}_k}\mathbf{A}'\boldsymbol{\Lambda}\mathbf{T}'_i\mathbf{R}^{-1}\boldsymbol{\omega}^{\star}_{\mathbf{b}_k i}\\ \boldsymbol{\Sigma}_{\mathbf{b}_k} &= \left(\mathbf{I} + \sum_{i\in\mathcal{S}_k}\mathbf{A}'\boldsymbol{\Lambda}\mathbf{T}'_i\mathbf{R}^{-1}\mathbf{T}_i\boldsymbol{\Lambda}\mathbf{A}\right)^{-1},\end{aligned}$$

and $\boldsymbol{\omega}^{\star}_{\mathbf{b}_k i} = \boldsymbol{\omega}_i - \mathbf{X}_i\boldsymbol{\beta}$.

Full conditional of v_{rd}: Under the Dirac spike, $\boldsymbol{v}$ should be sampled from its marginal posterior, which is obtained after integrating over $\boldsymbol{\beta}$; i.e.,

$$\begin{aligned}\pi(\boldsymbol{v} \mid \boldsymbol{\omega}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{\tau}_v) &\propto \pi(\boldsymbol{v}|\boldsymbol{\tau}_v)\int \pi(\mathbf{Z}, \widetilde{\mathbf{Y}}, \boldsymbol{\omega} \mid \boldsymbol{\Theta})\pi(\boldsymbol{\beta} \mid \boldsymbol{v})d\boldsymbol{\beta}\\ &\propto \pi(\boldsymbol{v}|\boldsymbol{\tau}_v)\pi(\boldsymbol{\omega} \mid \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}),\end{aligned}$$

where $\boldsymbol{\tau}_v = (\tau_{v_{rd}}; r = 1, ...p_d, , d = 1, ..., D)'$ and

$$\pi(\boldsymbol{\omega} \mid \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}) \propto |\boldsymbol{\Phi}(\boldsymbol{v})|^{-1/2}|\boldsymbol{\Sigma}_{\boldsymbol{\beta}}|^{1/2} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N} \boldsymbol{\omega}_{\boldsymbol{\beta}i}^{\star\prime}\mathbf{R}^{-1}\boldsymbol{\omega}_{\boldsymbol{\beta}i}^{\star} - \boldsymbol{\mu}_{\boldsymbol{\beta}}'\boldsymbol{\Sigma}_{\boldsymbol{\beta}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\beta}}\right]\right\}.$$

Here, $\boldsymbol{\Phi}(\boldsymbol{v})$, $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$, $\boldsymbol{\mu}_{\boldsymbol{\beta}}$, and $\boldsymbol{\omega}_{\boldsymbol{\beta}i}^{\star}$ are defined in the full conditional of $\boldsymbol{\beta}$ outlined above. It is worth noting that if $\boldsymbol{v} = \mathbf{0}$, then this marginalized likelihood reduces to $\exp\left\{-\frac{1}{2}\sum_{i=1}^{N} \boldsymbol{\omega}_{\boldsymbol{\beta}i}^{\star\prime}\mathbf{R}^{-1}\boldsymbol{\omega}_{\boldsymbol{\beta}i}^{\star}\right\}$. Thus, it is easy to see that the full conditional distribution of v_{rd}, after marginalizing over $\boldsymbol{\beta}$, is Bernoulli, with success probability $p_{v_{rd}}$; i.e., $v_{rd} \mid \boldsymbol{\omega}, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}_{(-rd)}, \tau_{v_{rd}} \sim \text{Bernoulli}(p_{v_{rd}})$, where $\boldsymbol{v}_{(-rd)}$ is the vector $\boldsymbol{v}$ after removing the rth element of $\boldsymbol{v}_d$ and

$$p_{v_{rd}} = \frac{\pi(\boldsymbol{\omega} \mid \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}_{(-rd)}, v_{rd} = 1)\tau_{v_{rd}}}{\pi(\boldsymbol{\omega} \mid \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}_{(-rd)}, v_{rd} = 0)(1 - \tau_{v_{rd}}) + \pi(\boldsymbol{\omega} \mid \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}, \mathbf{R}, \boldsymbol{v}_{(-rd)}, v_{rd} = 1)\tau_{v_{rd}}}.$$

Full conditional of w_{ld}: Under the Dirac spike, w_{ld} should be sampled from its marginal posterior, which is obtained after integrating over λ_{ld} the ℓth element of $\boldsymbol{\lambda}$; that is, sample from

$$\begin{aligned}\pi(w_{ld} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, \tau_{w_{ld}}) &\propto \pi(w_{ld}|\tau_{w_{ld}}) \int \pi(\mathbf{Z}, \widetilde{\mathbf{Y}}, \boldsymbol{\omega} \mid \boldsymbol{\Theta})\pi(\lambda_{ld} \mid w_{ld})d\lambda_{ld} \\ &\propto \pi(w_{ld}|\tau_{w_{ld}})\pi(\boldsymbol{\omega} \mid \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, w_{ld}),\end{aligned}$$

where $\boldsymbol{\lambda}_{(-\ell)}$ is the vector $\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}_{ld}$ removed and

$$\pi(\boldsymbol{\omega} \mid \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, w_{ld}) \propto \frac{\sigma_{\lambda_{ld}}(1 - \Phi(-\mu_{\lambda_{ld}}/\sigma_{\lambda_{ld}}))}{\psi_{ld}/2} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{N} \boldsymbol{\omega}_{\lambda_\ell i}^{\star\prime}\mathbf{R}^{-1}\boldsymbol{\omega}_{\lambda_\ell i}^{\star} - \mu_{\lambda_{ld}}^2/\sigma_{\lambda_{ld}}^2\right]\right\}.$$

Note, here all notational conventions developed to express the full conditional distribution of $\boldsymbol{\lambda}$ are adopted. Note that when $w_{ld} = 0$, then this marginalized likelihood reduces to $\exp\left\{-\frac{1}{2}\sum_{i=1}^{N} \boldsymbol{\omega}_{\lambda_\ell i}^{\star\prime}\mathbf{R}^{-1}\boldsymbol{\omega}_{\lambda_\ell i}^{\star}\right\}$. Thus, it is easy to see that the full conditional distribution of w_{ld}, after marginalizing over λ_{ld}, is Bernoulli, with probability $p_{w_{ld}}$; i.e., $w_{ld} \mid \boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, \tau_{w_{ld}} \sim \text{Bernoulli}(p_{w_{ld}})$, where

$$p_{w_{ld}} = \frac{\pi(\boldsymbol{\omega} \mid \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, w_{ld} = 1)\tau_{w_{ld}}}{\pi(\boldsymbol{\omega} \mid \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, w_{ld} = 0)(1 - \tau_{w_{ld}}) + \pi(\boldsymbol{\omega} \mid \boldsymbol{\beta}, \boldsymbol{\lambda}_{(-\ell)}, \mathbf{a}, \mathbf{b}, w_{ld} = 1)\tau_{w_{ld}}}.$$

Full conditionals of testing accuracies: The full conditionals for $S_{e(m):d}$ and $S_{p(m):d}$ are given by

$$\begin{aligned}S_{e(m):d} \mid \mathbf{Z}, \widetilde{\mathbf{Y}} &\sim \text{Beta}(a_{e(m):d}^{\star}, b_{e(m):d}^{\star}) \\ S_{p(m):d} \mid \mathbf{Z}, \widetilde{\mathbf{Y}} &\sim \text{Beta}(a_{p(m):d}^{\star}, b_{p(m):d}^{\star}),\end{aligned}$$

where

$$\begin{aligned}
a^{\star}_{e(m):d} &= a_{e(m):d} + \sum_{j \in \mathcal{I}_m} Z_{jd} \widetilde{Z}_{jd}, \\
b^{\star}_{e(m):d} &= b_{e(m):d} + \sum_{j \in \mathcal{I}_m} (1 - Z_{jd}) \widetilde{Z}_{jd}, \\
a^{\star}_{p(m):d} &= a_{p(m):d} + \sum_{j \in \mathcal{I}_m} (1 - Z_{jd})(1 - \widetilde{Z}_{jd}), \\
b^{\star}_{p(m):d} &= b_{p(m):d} + \sum_{j \in \mathcal{I}_m} Z_{jd}(1 - \widetilde{Z}_{jd}).
\end{aligned}$$