Coverage summary

We take a frequentist approach in comparing the coverage of the equal-tail and HPD empirical credible sets proposed in Section 2.3 to the large-sample Wald interval proposed by Bhattacharyya, Karandinos, and DeFoliart (1979). The Wald interval is typically used in practice and is given by $\hat{p}_{MLE} \pm z_{1-\alpha/2}\sqrt{v(\hat{p}_{MLE})/n}$ where $v(\hat{p}_{MLE}) = s^{-2}\left[1-(1-\hat{p}_{MLE})^{s}\right](1-\hat{p}_{MLE})^{2-s}$ and $z_{1-\alpha/2}$ denotes the $1 - \alpha/2$ quantile of the standard normal distribution. Since T follows a binomial distribution, the exact coverage is defined as

$$C(p,n,s) = \frac{\sum_{t=1}^{n-1} I(n,t,s)\binom{n}{t}\left[1-(1-p)^{s}\right]^{t}(1-p)^{s(n-t)}}{1-(1-p)^{sn}-\left[1-(1-p)^{s}\right]^{n}},$$

where $I(n,t,s) = 1$ if the interval contains p and $I(n,t,s) = 0$ otherwise. We did not consider the $t = 0$ and $t = n$ cases since the marginal maximum likelihood estimate for β can not be calculated.

Figure 1 displays the exact coverage ($\alpha = 0.05$) of the equal-tail and HPD credible intervals with $n = 40$, $s = 10$, and $p = 0.001$ to 0.10 by increments of 0.0005. A vertical line is given in the plots to indicate where $P(T = 0) = 0.01$ (p is greater than 0.10 for $P(T = n) = 0.01$). Both credible intervals use $\hat{\beta}_{MLE}$ as the marginal estimate of β. The intervals calculated with $\hat{\beta}_{MOM}$ are not displayed here since their coverage did not differ substantially from those intervals that used $\hat{\beta}_{MLE}$. For each HPD interval calculated, 10,000 samples from the posterior distribution are used in order to find the interval endpoints. We display the Wald interval coverage as a reference in both plots.

In terms of coverage probability, equal-tail and HPD credible intervals perform similar to and sometimes worse than the Wald interval. With regard to mean length (not shown), the Wald interval generally is wider. This same performance occurred for many other investigated settings of n, s, and p as well. Even though the Wald interval is used predominantly in practice, it still has two main drawbacks. First, the interval is symmetric, even though the true sampling distribution of $\hat{p}_{MLE}$ is not when n is small. Furthermore, symmetric confidence intervals are not preferred when p is close to zero. Second, Wald intervals can provide lower endpoints that are negative, which is clearly inappropriate. By their construction, our two empirical Bayes credible sets avoid both of these deficiencies (of course, credible sets and confidence intervals have different interpretations).

Additional references

Bhattacharyya, G., Karandinos, M., and DeFoliart, G., 1979: Point estimates and confidence intervals for infection rates using pooled organisms in epidemiological studies. *American Journal of Epidemiology* 109, 124-131.

Figure 1 Exact interval coverage when $\alpha = 0.05$, $n = 40$, and $s = 10$; the bold line is the plot title coverage and the non-bold line is the Wald coverage; the dashed vertical lines denote the p where $P(T = 0) = 0.01$.