**MA Models**

Moving average models – MA(q)

The white noise terms on the right side are linearly combined to form the model

xt = wt + θ1wt-1 + θ2wt-2 + … + θqwt-q = θ(B)wt

where

* θ1, θ2, …, θq are parameters,
* θ(B) = (1+θ1B+θ2B2+…+θqBq), and
* wt ~ independent (0,) for t = 1,…, n and typically assumed to have a normal distribution

Notes:

* “+” signs are used in the moving average operator, θ(B). “-“ signs were used in the autoregressive operator, ϕ(B). There is no particular reason why these are defined differently (other than what goes on the “right” side of the equality when writing the model out). I chose the way R and Shumway and Stoffer’s textbook defines these models. **Many** **books use “-“ signs also in θ(B), so be careful!** This will cause items like ACFs to be represented differently.
* The autocovariance of a MA(1):

xt = wt + θ1wt-1 where wt ~ ind. (0,) for t = 1, …, n

γ(h) = Cov(xt, xt+h) = E(xtxt+h) – E(xt)E(xt+h)

= E(xtxt+h) - 0 = E(xtxt+h)

Then

γ(h) = E[(wt + θ1wt-1)(wt+h + θ1wt+h-1)]

= E[wtwt+h + θ1wtwt+h-1 + θ1wt-1wt+h + wt-1wt+h-1]

= E[wtwt+h] + θ1E[wtwt+h-1] + θ1E[wt-1wt+h]

+ E[wt-1wt+h-1]

For h = 0:

E[] + θ1E[wtwt-1] + θ1E[wt-1wt] + E[]

= Var(wt) + [E(wt)]2 + 2θ1E[wt]E[wt-1] + Var(wt-1)

+ [E(wt-1)]2

= + 02 + 2θ1×0×0 + + 02

= (1+)

For h = 1:

E[wtwt+1] + θ1E[] + θ1E[wt-1wt+1] + E[wt-1wt]

= E[wt]E[wt+1] + θ1{Var(wt) + [E(wt)]2}

+ θ1E[wt-1]E[wt+1] + E[wt-1]E[wt]

= 0×0 + θ1{ +02} + θ1×0×0 + ×0×0

= θ1

For h = 2:

E[wtwt+2] + θ1E[wtwt+1] + θ1E[wt-1wt+2]

+ E[wt-1wt+1] = 0

For h > 2: γ(h) = 0.

Therefore,



* The autocorrelation function of a MA(1):



Notice the autocorrelation is 0 for h > 1!!!

* We have already seen an “infinite order” (q = ) moving average process as defined in the form of a linear process. This came up in our AR(p) examples earlier. Again, a linear process can be defined as

 with  and   
 wt ~ ind. N(0,)

where it can be shown that  for h ≥ 0. For a MA(1), ψ0 = 1 and ψ1 = θ1 and the remaining ψj’s equal to 0. This results in



and

,



 for h ≥ 2

Example: MA(1) with θ1 = 0.7 and –0.7 (ma1\_sim.R)

The purpose of this example is to show what observed values from a MA(1) process look like for t = 1, …, 100. Pay close attention to the differences between θ1 = 0.7 and θ1 = -0.7. Questions to think about are:

* Why are some plot plots more or less “choppy” (“jagged”)?
* What would happen to the plots if |θ1| was closer to 0 or 1?

The autocorrelations are

|  |  |  |
| --- | --- | --- |
| h | θ1 = 0.7 | θ1 = -0.7 |
| 1 | 0.47 | -0.47 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
|  |  |  |





R Code (use ma = c(-0.7) for the second plot):

> set.seed(8199)

> x <- arima.sim(model = list(ma = c(0.7)), n = 100,

rand.gen = rnorm, sd = 10)

> plot(x = x, ylab = expression(x[t]), xlab = "t", type =

"l", col = c("red"), main =

expression(paste(x[t] == w[t] + 0.7\*w[t-1], " where ",

w[t], " ~ ind. N(0,100)")) , panel.first=grid(col =

"gray", lty = "dotted"))

> points(x = x, pch = 20, col = "blue")

Using ARMAacf()results in,

> par(mfrow = c(1,2))

> round(ARMAacf(ma = c(0.7), lag.max = 20),4)

0 1 2 3 4 5 6 7 8 9 10

1.0000 0.4698 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

11 12 13 14 15 16 17 18 19 20

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

> plot(y = ARMAacf(ma = c(0.7), lag.max = 20), x = 0:20

type = "h", ylim = c(-1,1), xlab = "h", ylab =

expression(rho(h)), main = expression(paste("ACF for

MA(1) with ", theta[1] == 0.7)))

> abline(h = 0)

> round(ARMAtoMA(ma = c(0.7), lag.max = 5),4)

[1] 0.7 0.0 0.0 0.0 0.0

> plot(y = ARMAacf(ma = c(-0.7), lag.max = 20), x =

0:20, type = "h", ylim = c(-1,1), xlab = "h", ylab =

expression(rho(h)), main = expression(paste("ACF for

MA(1) with ", theta[1] == -0.7)))

> abline(h = 0)



Using ARMAtoMA()produces the obvious result.

> round(ARMAtoMA(ma = c(0.7), lag.max = 5),4)

[1] 0.7 0.0 0.0 0.0 0.0

> #Example for MA(2)

> round(ARMAtoMA(ma = c(0.7, -0.4), lag.max = 5),4)

[1] 0.7 -0.4 0.0 0.0 0.0

Invertible process

A MA(q) process can be represented as an infinite order AR process. Thus,



where π(B) = 1+π1B+π2B2+ … . This is “similar” to how an AR(p) process can be represented as an infinite order MA process.

A MA process can have the same autocorrelation for multiple values of θ and . For example, consider the models

xt = wt + 0.1wt-1

and

xt = wt + 10wt-1

The autocorrelation at lag 1 is ρ(1) = 0.09. In fact, this occurs for any MA(1) model using θ1 and 1/θ1 as the coefficient on wt-1.

The model that is chosen to represent a series is the one that has an infinite order AR representation. This chosen process is called an “invertible” process.

A more formal definition of an invertible process will be given later (similar to causal process requirements).